International Journal of Thermophysics, Vol. 24, No. 1, January 2003 (© 2003)

Transient Disturbance in a Half Space Under Thermoelasticity with Two Relaxation Times Due to Moving Internal Heat Source

Y. Q. Song,[1,2] **Y. C. Zhang,**[1] **and B. H. Lu**[3]

Received November 16, 2001

In this paper transient waves caused by a line heat source moving with a uniform velocity inside isotropic homogeneous thermoelastic half-space are studied under the GL model of generalized thermoelasticity. The problem is reduced to the solution of three differential equations by introducing the elastic vector potential and the thermoelastic scalar potential. Using Laplace and Fourier transforms solutions are obtained in transforms domain. Applying inverse transforms approximate solutions of the displacement at the boundary valid in the small time range are given. Also the approximate region valid for the solutions is given and two special cases, (i) the source is motionless and (ii) the relaxation times vanish, are studied. Numerical evaluations are presented for the medium of copper.

KEY WORDS: GL theory; moving source; thermal wave.

1. INTRODUCTION

The classical linear theory of thermoelasticity, which is based in part on the assumption that the heat flux vector satisfies Fourier heat conduction law, predicts finite propagation speeds for elastic waves but an infinite speed for thermal disturbance. This is physically unrealistic. The generalized thermoelastic theory proposed by Lord and Shulman [1] and Green and Lindsay [2] (here called LS and GL theories, respectively) have aroused much

[1] Department of Engineering Mechanics, Xi'an Jiaotong University, Xi'an 710049, People's Republic of China.
[2] To whom correspondence should be addressed. E-mail: mssvvip@xjtu.edu.cm
[3] Department of Mechanical Engineering, Xi'an Jiaotong University, Xi'an 710049, People's Republic of China.

0195-928X/03/0100-0299/0

interest in recent years. The LS theory introduced a single time constant to dictate the relaxation of thermal propagation. This theory is based on a new law of heat conduction to replace Fourier's law. The heat equation is a hyperbolic one that ensures finite speed of propagation for heat and elastic waves. In the GL theory, thermal and thermalmechanical relaxation are governed by two different time constants and the temperature rates are considered among the constitutive variables. This model admit second sound even without violating the classical Fourier's law. The two theories are structurally different from one other, and one cannot be obtained as a particular case of the other. Various problems characterizing these two theories have been investigated, and reveal some interesting phenomena. Brief reviews of this topic have been reported by Chandrasekharaiah [3, 4].

Heat source acting in an elastic body is widely used in engineering involving materials processing, case hardening and boiling nucleation etc. This kind of problem is very interesting in mathematic and important in physics. Sarbani and Amitava [5] studied the transient disturbances in half-space due to moving internal heat source under LS model and obtained the solution for the displacements in the transform domain. Under GL theory, Chandrasekharaiah and Srikantiah [6] have studied the cases of both continuous and impulsive point heat sources in an unbounded body and obtained small time solutions with the aid of Laplace transforms. Chandrasekharaiah and Murthy [7] have studied cylindrical waves due to a continuous line heat source in an unbounded body and have obtained small-time solutions by employing the Laplace and Hankel transform. Ignaczak and Mrowka-Matejewska [8] have considered one-dimensional waves produced by an infinite body and have presented a closed-form solution for the displacement-heat flux formulation of the problem. Hetnarski and Ignaczak [9] have studied a plane heat sources in a half-space in great detail and have presented closed form solutions.

In this paper, transient waves created by a line heat source that suddenly starts moving with a uniform velocity inside isotropic homogeneous thermoelastic half-space are studied with thermal relaxation of the type of Green and Lindsay. The source moves parallel to the boundary surface is stress free. The problem is reduced to the solution of three differential equations, one involving the elastic vector potential, and the other two coupled, involving the thermoelastic scalar potential and the temperature. The problem is solved using Laplace and Fourier transforms. The expression for displacements valid in the small time range are obtained in transforms domain and the displacements are calculated on the boundary by using inverse transforms for small time. The approximate region valid for the solution is given and two special cases are considered. Also the results are graphically described for the medium of copper.

2. FORMULATION OF THE PROBLEM

Consider a linear, homogenous and isotropic thermoelastic continuum occupying the region $x_2 \geqslant -h$ which is in a quiescent state, and the surface $x_2 \geqslant -h$ is stress free. A line source suddenly starts moving inside the medium at a depth h below the free surface with a uniform velocity in the x_1 direction. The line source is parallel to the x_3 axis so that all quantities are independent of x_3, and the third component u_3 of the displacement vector vanishes. When all body forces are neglected the governing equations are:

(1) Strain-displacement relation:

$$2\tau_{ij} = u_{i,j} + u_{j,i}, \qquad i, j = 1, 2 \tag{1}$$

where u_i $(i = 1, 2)$ is the component of displacement vector, τ_{ij} $(i, j = 1, 2)$ is the component of strain tensor.

(2) Stress-displacement relation:

$$\sigma_{ij} = \mu(u_{i,j} + u_{j,i}) + \lambda u_{k,k}\delta_{ij} - \gamma\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\theta\delta_{ij}, \qquad i, j = 1, 2 \tag{2}$$

where σ_{ij} $(i, j = 1, 2)$ is the component of stress tensor, λ, μ are Lame's constants, θ is the absolute temperature, τ_1 is a relaxation constant with the dimensions of time and $\gamma = (3\lambda + 2\mu)\,\alpha_t$, with α_t being the coefficient of linear thermal expansion.

(3) Heat conduction equation:

$$\rho c\left(1 + \tau_2 \frac{\partial}{\partial t}\right)\dot{\theta} + \theta_0\gamma\dot{u}_{i,i} - Q = k\theta_{,ii}, \qquad i = 1, 2 \tag{3}$$

where ρ is mass density, c the specific heat at constant strain, k is the thermal conductivity, τ_2 is another relaxation time and θ_0 is a reference temperature and Q is heat source.

(4) Equation of motion

$$\mu u_{i,jj} + (\lambda + \mu)\, u_{j,ij} - \gamma\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\theta_{,i} = \rho\ddot{u}_i, \qquad i, j = 1, 2 \tag{4}$$

where parameters τ_1, τ_2 are characteristic of this theory, and in order to hold the entropy production inequality, they must satisfy the inequality $\tau_1 \geqslant \tau_2 \geqslant 0$. If $\tau_1 = \tau_2 = 0$, this theory reduces to classical coupled thermoelasticity.

(5) Initial conditions and boundary conditions:

Initial conditions are

$$\left.\begin{array}{llll} u_i = 0, \theta = 0, & \text{at} & t \leqslant 0 & \text{in } x_2 \geqslant -h \\ \dot{u}_i = 0, \dot{\theta} = 0, & \text{at} & t \leqslant 0 & \text{in } x_2 \geqslant -h \end{array}\right\} \quad i = 1, 2 \tag{5}$$

The stress-free boundary conditions are

$$\sigma_{12} = \sigma_{22} = 0 \quad \text{on} \quad x_2 = -h \quad \text{for} \quad t \geqslant 0 \tag{6}$$

The regularity conditions are

$$\theta, u_i \to 0 \quad \text{as} \quad x_2 \to \infty, \quad x_1 \to \pm\infty \tag{7}$$

The thermal boundary condition at $x_2 = -h$ to be imposed is

$$\theta = 0 \tag{8}$$

Introducing the scalar and vector potentials ϕ, $(0, 0, \psi)$ and defined by:

$$\begin{cases} u_1 = \phi_{,1} + \psi_{,2} \\ u_2 = \phi_{,2} - \psi_{,1} \end{cases} \tag{9}$$

where

$$\phi = \phi(x_1, x_2, t), \quad \text{and} \quad \psi = \psi(x_1, x_2, t). \tag{10}$$

Taking divergence and curl of equation of motion gives

$$\nabla^2\phi - m_1\theta - \tau_1 m_1 \dot{\theta} = \frac{1}{C_L^2}\ddot{\phi} \tag{11}$$

$$\nabla^2\psi = \frac{1}{C_T^2}\ddot{\psi} \tag{12}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad m_1 = \frac{\gamma}{\lambda + 2\mu}, \quad C_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad C_T = \sqrt{\frac{\mu}{\rho}}$$

From Eq. (3) we can obtain

$$k\nabla^2\theta = \rho c\left(1 + \tau_2 \frac{\partial}{\partial t}\right)\dot{\theta} + \gamma\theta_0 \nabla^2\dot{\phi} - Q \tag{13}$$

In order to nondimensionalize Eqs. (11)–(13), let us define the following set of dimensionless variables:

$$\bar{x}_i = \frac{x_i}{C_T\omega}, \quad \bar{u}_i = \frac{u_i}{C_T\omega}, \quad \bar{\theta} = m_1\theta, \quad \bar{\phi} = \frac{\phi}{(C_T\omega)^2}, \quad \bar{\psi} = \frac{\psi}{(C_T\omega)^2}, \quad \bar{t} = \frac{t}{\omega}, \tag{14}$$

$$\bar{\tau}_1 = \frac{\tau_1}{\omega}, \quad \bar{\tau}_2 = \frac{\tau_2}{\omega}, \quad Q = \frac{m_1\omega Q}{\rho c}, \quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\mu},$$

where

$$\omega = \frac{k}{\rho c C_T^2} \tag{15}$$

Equations (11)–(13) become

$$\bar{\nabla}^2\bar{\phi} - \bar{\theta} - \bar{\tau}_1\dot{\bar{\theta}} = \frac{1}{\beta^2}\ddot{\bar{\phi}} \tag{16}$$

$$\bar{\nabla}^2\bar{\psi} = \ddot{\bar{\psi}} \tag{17}$$

$$\bar{\nabla}^2\bar{\theta} = (\bar{\theta} + \dot{\bar{\tau}}_2\ddot{\bar{\theta}}) + \bar{\varepsilon}\,(\bar{\nabla}^2\dot{\bar{\phi}}) - \bar{Q} \tag{18}$$

where

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial\bar{x}_1^2} + \frac{\partial^2}{\partial\bar{x}_2^2}, \qquad \beta = \frac{C_L}{C_T}, \qquad \bar{\varepsilon} = \frac{\gamma\theta_0 m_1}{\rho c} \tag{19}$$

and ε is the coupling parameter. The bar over the quantity will be omitted below for convenience. Also we shall write x, y for x_1, x_2 and u, v for u_1, u_2 respectively.

The Laplace transform of a function $A(x, y, t)$ with respect to t is defined as

$$L(A(x, y, t)) = \int_0^\infty e^{-pt}A(x, y, t)\,dt = \tilde{A}(x, y, p)$$

and the Fourier transform of $\tilde{A}$ with respect to x is defined as

$$F(\tilde{A}(x, .y, p)) = \int_{-\infty}^{\infty} e^{-i\xi x}\tilde{A}(x, y, p)\,dx = A^*(\xi, .y, p)$$

Inverting the transforms give

$$A(x, y, t) = L^{-1}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} A^*(\xi, y, p)\, e^{i\xi x}\, d\xi\right\}$$

where L^{-1} denote the inverse Laplace transform.

Applying Laplace and Fourier transforms to Eqs. (16)–(18) and using the initial conditions given in Eqs. (5), we have

$$\left(\frac{d^2}{dy^2}-\xi^2-\frac{p^2}{\beta^2}\right)\phi^* = \theta^* + \tau_1 p\theta^* \tag{20}$$

$$\left(\frac{d^2}{dy^2}-\xi^2-p^2\right)\psi^* = 0 \tag{21}$$

$$\left(\frac{d^2}{dy^2}-\xi^2-p(1+p\tau_2)\right)\theta^* - \varepsilon p\left(\frac{d^2}{dy^2}-\xi^2\right)\phi^* = -Q^* \tag{22}$$

The moving source is located at the origin, at time $t = 0^+$ and starts moving along the positive x-axis, with uniform velocity V. The source is assumed in the form (the same as in [5])

$$Q = Q_0\delta(x-Vt)\,\delta(y)\,H(t), \qquad Q^* = Q_0\delta(y)/(p+i\xi V). \tag{23}$$

From Eq. (19) we can obtain

$$\theta^* = \frac{1}{1+\tau_1 p}\left(\frac{d^2}{dy^2}-\xi^2-\frac{p^2}{\beta^2}\right)\phi^* \tag{24}$$

3. SOLUTIONS OF THE PROBLEM IN TRANSFORMS DOMAIN

Eliminating θ^* from Eqs. (20) and (22), we get a fourth-order differential equation in ϕ^*

$$\left[D^2-\left(\frac{p^2}{\beta^2}+p(1+p\tau_2+\varepsilon(1+p\tau_1))\right)D+\frac{p^3(1+p\tau_2)}{\beta^2}\right]\phi^* = -\frac{Q_0(1+p\tau_1)\,\delta(y)}{p+i\xi V} \tag{25}$$

where

$$D \equiv \frac{d^2}{dy^2}-\xi^2 \tag{26}$$

The solution for ϕ^*, under the regularity condition is given by

$$\phi^* = \begin{cases} A_1 e^{-a_1 y} + A_2 e^{-a_2 y} + A_3 e^{a_1 y} + A_4 e^{a_2 y}, & -h < y < 0 \\ B_1 e^{-a_1 y} + B_2 e^{-a_2 y}, & y > 0 \end{cases} \tag{27}$$

The solution of Eq. (21) satisfying regularity conditions is

$$\psi^* = \begin{cases} C_1 e^{-(\xi^2+p^2)^{1/2} y} + C_2 e^{(\xi^2+p^2)^{1/2} y}, & -h < y < 0 \\ D e^{-(\xi^2+p^2)^{1/2} y}, & y > 0 \end{cases} \tag{28}$$

where

$$\begin{aligned} a_{1,2}^2 &= \xi^2 + \frac{1}{2}\left[\frac{p^2}{\beta^2} + p(1 + p\tau_2 + \varepsilon(1 + p\tau_1))\right] \\ &\pm \frac{1}{2}\left[\left(\frac{p^2}{\beta^2} + p(1 + p\tau_2 + \varepsilon(1 + p\tau_1))\right)^2 - \frac{4p^3(1 + p\tau_2)}{\beta^2}\right]^{\frac{1}{2}} \end{aligned} \tag{29}$$

Both $a_{1,2}$ are assumed to be real and positive.

The displacement fields u^*, v^* are

$$u^* = \begin{cases} i\xi(A_1 e^{-a_1 y} + A_2 e^{-a_2 y} + A_3 e^{a_1 y} + A_4 e^{a_2 y}) - C_1 b_1 e^{-b_1 y} + C_2 b_1 e^{b_1 y}, \\ \qquad -h < y < 0 \\ i\xi(B_1 e^{-a_1 y} + B_2 e^{-a_2 y}) - D b_1 e^{-b_1 y}, \qquad y > 0 \end{cases} \tag{30}$$

$$v^* = \begin{cases} -a_1 A_1 e^{-a_1 y} - a_2 A_2 e^{-a_2 y} + a_1 A_3 e^{a_1 y} + a_2 A_4 e^{a_2 y} - i\xi(C_1 e^{-b_1 y} + C_2 e^{b_1 y}), \\ \qquad -h < y < 0 \\ -a_1 B_1 e^{-a_1 y} - a_2 B_2 e^{-a_2 y} - i\xi D e^{-b_1 y}, \qquad y > 0 \end{cases} \tag{31}$$

where

$$b_1 = (\xi^2 + p^2)^{1/2} \tag{32}$$

The stresses are transformed as

$$\sigma_{xx}^* = p^2\phi^* - 2\frac{d^2\phi}{dy^2} + 2i\xi\frac{d\psi^*}{dy} \tag{33}$$

$$\sigma_{yy}^* = (p^2 + 2\xi^2)\,\phi^* - 2i\xi\frac{d\psi^*}{dy} \tag{34}$$

$$\sigma_{xy}^* = 2i\xi\frac{d\phi^*}{dy} + \frac{d^2\psi^*}{dy^2} + \xi^2\psi^* \tag{35}$$

Moreover from the stress free boundary conditions, we have

$$\sigma^*_{xy} = \sigma^*_{yy} = 0 \qquad \text{on} \quad y = -h \tag{36}$$

Further, since the stress components are continuous across $y = 0$, it follows that

$$\phi^*, \quad \frac{d\phi^*}{dy}, \quad \frac{d^2\phi^*}{dy^2}, \quad \psi^*, \quad \frac{d\psi^*}{dy}, \quad \frac{d^2\psi^*}{dy^2} \tag{37}$$

are all continuous across $y = 0$.

To obtained the jump discontinuity due to the presence of $\delta(y)$ in Q^*, Eq. (22) is integrated from $y = -\eta$ to $y = \eta$, $(\eta > 0)$ and finally made η to tend to 0^+

$$\left.\frac{d^3\phi^*}{dy^3}\right|_{y\to 0^\pm} = \left.\frac{d\theta^*}{dy}\right|_{y\to 0^\pm} = -\frac{Q_0(1+p\tau_1)}{p+i\xi V} \tag{38}$$

Using conditions (37), (38), we obtain a set of seven equations involving the nine constants A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2, D

$$A_1 + A_2 + A_3 + A_4 - B_1 - B_2 = 0 \tag{39}$$

$$A_1 a_1 + A_2 a_2 - A_3 a_1 - A_4 a_2 - B_1 a_1 - B_2 a_2 = 0 \tag{40}$$

$$A_1 a_1^2 + A_2 a_2^2 + A_3 a_1^2 + A_4 a_2^2 - B_1 a_1^2 - B_2 a_2^2 = 0 \tag{41}$$

$$C_1 + C_2 - D = 0 \tag{42}$$

$$-C_1 b_1 + C_2 b_1 + D b_1 = 0 \tag{43}$$

$$C_1 b_1^2 + C_2 b_1^2 - D b_1^2 = 0 \tag{44}$$

$$-A_1 a_1^3 - A_2 a_2^3 + A_3 a_1^3 + A_4 a_2^3 + B_1 a_1^3 + B_2 a_2^3 = \frac{Q_0(1+p\tau_1)}{p+i\xi V} \tag{45}$$

From equations above we can see that the Eq. (42) is the same as Eq. (44), so there has six absolute equations.

The condition (36) imply

$$d(A_1 e^{a_1 h} + A_2 e^{a_2 h} + A_3 e^{-a_1 h} + A_4 e^{-a_2 h}) + 2i\xi b_1 C_1 e^{b_1 h} + 2i\xi b_1 C_2 e^{b_1 h} = 0 \tag{46}$$

$$2i\xi(-a_1 A_1 e^{a_1 h} - a_2 A_2 e^{a_2 h} + a_1 A_3 e^{-a_1 h} + a_2 A_4 e^{-a_2 h}) + dC_1 e^{b_1 h} + dC_2 e^{b_1 h} = 0 \tag{47}$$

where

$$d = p^2 + 2\xi^2 \tag{48}$$

using Eq. (24), condition (8) reduces to

$$A_1(a_1^2 - b_2)\, e^{a_1 h} + A_2(a_2^2 - b_2)\, e^{a_2 h} + A_3(a_1^2 - b_2)\, e^{-a_1 h} + A_4(a_2^2 - b_2)\, e^{-a_2 h} = 0 \tag{49}$$

where

$$b_2 = \xi^2 + (p^2/\beta^2) \tag{50}$$

From Eqs. (39)–(47) and (49), we obtain

$$\begin{aligned} A_1 &= -K\{e^{(a_2-a_1)h}(a_1+a_2)[d^2(a_1-a_2) - 4\xi^2 b_1 a_1 a_2 + 4\xi^2 b_1 b_2] \\ &\quad + 8a_1 b_1 (a_2^2 - b_2)\}/a_1 \Delta e^{(a_1+a_2)h} \end{aligned} \tag{51}$$

$$\begin{aligned} A_2 &= -K\{e^{(a_2-a_1)h}(a_1+a_2)[d^2(a_1-a_2) - 4\xi^2 b_1 a_1 a_2 + 4\xi^2 b_1 b_2] \\ &\quad + 8a_2 b_1 (a_1^2 - b_2)\}/a_2 \Delta e^{(a_1+a_2)h} \end{aligned} \tag{52}$$

$$A_3 = \frac{K}{2a_1}, \qquad A_4 = -\frac{K}{2a_2} \tag{53}$$

$$B_1 = A_1 + A_3, \qquad B_2 = A_2 + A_4 \tag{54}$$

$$C_2 = 0, \qquad C_1 = D = ide^{-b_1 h}(A_1 e^{a_1 h} + A_2 e^{a_2 h} + A_3 e^{-a_1 h} + A_4 e^{-a_2 h})/2\xi b_1 \tag{55}$$

where

$$K = \frac{Q_0(1+p\tau_1)}{2(a_1^2 - a_2^2)(p + i\xi V)} \tag{56}$$

$$\Delta = (a_1 - a_2)\{d^2(a_1+a_2) - 4\xi^2 b_1 a_1 a_2 - 4\xi^2 b_1 b_2\} \tag{57}$$

The surface displacements are

$$u^*|_{y=-h} = \frac{2ip^2 b_1 \xi Q_0(1+p\tau_1)}{(p+i\xi V)\,\Delta}\,(e^{-a_1 h} - e^{-a_2 h}), \tag{58}$$

$$v^*|_{y=-h} = \frac{p^2 d Q_0(1+p\tau_1)}{(p+i\xi V)\,\Delta}\,(e^{-a_1 h} - e^{-a_2 h}). \tag{59}$$

4. SOLUTIONS VALID FOR SMALL TIME REGION

To the first order of approximation in ε, a_1^2 and a_2^2 may be written as

$$a_1^2 = \xi^2 + \frac{p^2}{\beta^2} - \frac{p^2(1+p\tau_1)}{\beta^2(1+p\tau_3)}\,\varepsilon \tag{60}$$

$$a_2^2 = \xi^2 + p + p^2\tau_2 + \frac{p(1+p\tau_1)(1+p\tau_2)}{1+p\tau_3}\,\varepsilon \tag{61}$$

where

$$\tau_3 = \tau_2 - \frac{1}{\beta^2} \tag{62}$$

For a short time approximation to the displacement components, we expand a_1, a_2, b_1 in terms of power of p, and consider relevant terms as $p \to \infty$. Then

$$a_1 \sim \frac{pn_1}{\beta} + \frac{n_3\varepsilon}{2\beta\tau_3^2 n_1} + \frac{4\beta^2\xi^2\tau_3^2 + n_3\varepsilon^2(3\tau_1+\tau_3) - 4\varepsilon\tau_3(n_3+\beta^2\xi^2\tau_1\tau_3^2)}{8\beta\tau_3^4 p n_1^3} \tag{63}$$

$$a_2 \sim p\sqrt{\tau_2}\,n_2 + \frac{\tau_3^2 + n_4\varepsilon}{2\tau_3^2\sqrt{\tau_2}\,n_2} + \frac{4\tau_2\tau_3[\tau_3^3\xi^2 - (n_4 - \tau_3^2)\,\varepsilon] - (\tau_3^2 + n_4\varepsilon)^2}{8p\tau_2\sqrt{\tau_2}\,n_2\tau_3^4} \tag{64}$$

$$b_1 \sim p + \frac{\xi^2}{2p} \tag{65}$$

where

$$n_1 = \sqrt{1 - \frac{\tau_1\varepsilon}{\tau_3}}, \quad n_2 = \sqrt{1 + \frac{\tau_1\varepsilon}{\tau_3}}, \quad n_3 = \tau_1 - \tau_3, \quad n_4 = \tau_1\tau_3 + \tau_2\tau_3 - \tau_1\tau_2 \tag{66}$$

It is clear from Eqs. (63)–(65) that there are three waves with velocities β/n_1, $1/\sqrt{\tau_2}\,n_2$, 1 respectively representing the dilatational waves, the thermalelastic waves and the transverse elastic waves.

For a small time, using Eqs. (63)–(65) in (58), (59)

$$u^*|_{y=-h} \sim \frac{2iQ_0\xi(1+p\tau_1)(e^{-a_1h} - e^{-a_2h})}{(p+i\xi V)\,p^3(L_1^2 + 4\xi^2L_2^2)} \tag{67}$$

$$v^*|_{y=-h} \sim \frac{Q_0(1+p\tau_1)(e^{-a_1h} - e^{-a_2h})}{(p+i\xi V)\,p^2(L_1^2 + 4\xi^2L_2^2)} \tag{68}$$

where

$$L_1^2 = \sqrt{\left(\frac{1}{\beta^2}+\tau_2+\tau_1\varepsilon\right)^2 - \frac{4\tau_2}{\beta^2}}, \qquad L_2^2 = \frac{n_1-\beta n_2\sqrt{\tau_2}}{\beta^5} \tag{69}$$

Finally, using software (MS Mathematica 4.0) we can take inverse Laplace and Fourier transform from Eqs. (67) and (68), gives

$$u|_{\substack{y=-h \\ x\geqslant 0}} \approx -\frac{\sqrt{2\pi}\,Q_0}{4L_1^3L_2V^3}\left[e^{-c_2h}H(t-c_1h)\,f_1 - e^{-c_4h}H(t-c_3h)\,f_2\right] \tag{70}$$

$$u|_{\substack{y=-h \\ x<0}} \approx -\frac{\sqrt{2\pi}\,Q_0}{4L_1^3L_2V^3}\left\{e^{-c_2h}\left[H\left(t-c_1h+\frac{x}{V}\right)f_3 + H(t-c_1h)\,f_5\right]\right.$$
$$\left. - e^{-c_4h}\left[H\left(t-c_3h+\frac{x}{V}\right)f_4 + H(t-c_3h)\,f_6\right]\right\} \tag{71}$$

$$v|_{\substack{y=-h \\ x\geqslant 0}} \approx \frac{\sqrt{2\pi}\,Q_0}{2L_1^3V^2}\left[e^{-c_2h}H(t-c_1h)\,f_7 - e^{-c_4h}H(t-c_3h)\,f_8\right] \tag{72}$$

$$v|_{\substack{y=-h \\ x<0}} \approx \frac{\sqrt{2\pi}\,Q_0}{2L_1^3V^2}\left\{e^{-c_2h}\left[H\left(t+\frac{x}{V}-c_1h\right)f_9 + H(t-c_1h)\,f_{11}\right]\right.$$
$$\left. - e^{-c_4h}\left[H\left(t+\frac{x}{V}-c_3h\right)f_{10} + H(t-c_3h)\,f_{12}\right]\right\} \tag{73}$$

where

$$c_1 = \frac{n_1}{\beta}, \quad c_2 = \frac{n_3\varepsilon}{2\beta\tau_3^2 n_1}, \quad c_3 = \sqrt{\tau_2}\,n_2, \quad c_4 = \frac{\tau_3^2+n_4\varepsilon}{2\tau_3^2\sqrt{\tau_2}\,n_2} \tag{74}$$

$$f_{1(2)} = -4L_2(2L_2+L_1V\tau_1)\,e^{\frac{L_1(c_{1(3)}hV-tV-x)}{2L_2}} + [8L_2^2 - 4L_1L_2V(c_{1(3)}h-t-\tau_1)$$
$$+ L_1^2V^2(c_{1(3)}h-t)(c_{1(3)}h-t-2\tau_1)]\,e^{-\frac{L_1x}{2L_2}} \tag{75}$$

$$f_{3(4)} = 4L_2(-2L_2+L_1V\tau_1)\,e^{\frac{L_1(c_{1(3)}hV-tV-x)}{2L_2}} + 4L_2(2L_2+L_1V\tau_1)\,e^{-\frac{L_1(c_{1(3)}hV-tV-x)}{2L_2}}$$
$$+ 8L_1L_2(c_{1(3)}hV - tV - V\tau_1 - x) \tag{76}$$

$$f_{5(6)} = -4L_2(2L_2+L_1V\tau_1)\,e^{-\frac{L_1(c_{1(3)}hC-tV-x)}{2L_2}} + [8L_2^2 - 4L_1L_2V(c_{1(3)}h-t-\tau_1)$$
$$+ L_1^2V^2(c_{1(3)}h-t)(c_{1(3)}h-t-2\tau_1)]\,e^{\frac{L_1x}{2L_2}} \tag{77}$$

$$f_{7(8)} = (-2L_2 + L_1 V \tau_1)\, e^{\frac{L_1 V (c_{1(3)} hV - tV - x)}{2L_2}} + [2L_2 + L_1 V (c_{1(3)} h - t - \tau_1)]\, e^{-\frac{L_1 x}{2L_2}} \tag{78}$$

$$f_{9(10)} = (2L_2 + L_1 V \tau_1)\, e^{-\frac{L_1 (c_{1(3)} hV - tV - x)}{2L_2}} - (2L_2 - L_1 V \tau_1)\, e^{\frac{L_1 (c_{1(3)} hV - tV - x)}{2L_2}} + 2L_1 (c_1 hV - tV - V\tau_1 - x) \tag{79}$$

$$f_{11(12)} = -(2L_2 + L_1 V \tau_1)\, e^{-\frac{L_1 (c_{1(3)} hV - tV - x)}{2L_2}} + [2L_2 - L_1 V (c_{1(3)} h - t - \tau_1)]\, e^{\frac{L_1 x}{2L_2}} \tag{80}$$

It is clear that the surface displacements for small time consist of dilatational waves propagating with velocity $(1/c_1)$ and a thermalelastic wave moving with velocity $(1/c_3)$. Also the waves are attenuated by exponential factors depending on ε, τ_1 and τ_2.

If the velocity of heat source $V \to 0$, the displacements can be obtained using the laws of L'Hospital limits. Here $x/V \to \infty$ results in $H(t - c_{1(3)} h + x/V) \to 0$. The displacement expressions (70)–(73) tends to the expressions below when V approaches the limit 0:

$$u|_{y=-h} \to \frac{\sqrt{2\pi}\, Q_0}{24 L_2^2} [e^{-c_2 h} f_{13} H(t - c_1 h) - e^{-c_4 h} f_{14} H(t - c_3 h)] \tag{81}$$

$$v|_{y=-h} \to \frac{\sqrt{2\pi}\, Q_0}{8 L_1 L_2} [e^{-c_2 h} H(t - c_1 h)\, f_{15} - e^{-c_4 h} H(t - c_3 h)\, f_{16}] \tag{82}$$

where

$$f_{13(14)} = (t - c_{1(3)} h)^2 (c_{1(3)} h - t - 3\tau_1) \left[-\sinh\left(\frac{L_1 x}{2L_2}\right) + \cosh\left(\frac{L_1 x}{2L_2}\right) \mathrm{Sign}[x] \right] \tag{83}$$

$$f_{15(16)} = (t - c_{1(3)} h)(t - c_{1(3)} h + 2\tau_1) \left[-\cosh\left(\frac{L_1 x}{2L_2}\right) + \sinh\left(\frac{L_1 x}{2L_2}\right) \mathrm{Sign}[x] \right] \tag{84}$$

and

$$\mathrm{Sign}[x] = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 1 \end{cases} \tag{85}$$

Obviously, the x-component of the displacement is antisymmetric with respect to x and the y-component of the displacement is symmetrical with respect to x.

5. ANALYSIS OF THE RESULTS

Here we are interested in the displacements at the boundary. From Eqs. (63)–(65) we can see there must be three waves representing the dilatational waves, the thermalelastic waves and the transverse elastic waves respectively. But we can see from the Eqs. (70)–(73) that there only exist two waves at the boundary. The third solution must be zero at the boundary in order to satisfy the thermal boundary condition and the stress-free boundary conditions.

We consider the short-time approximation solution of the problem. However, the time should be long enough to waves reach the boundary. Also, the more remote points at the boundary require longer times to reach them, therefore the approximate solutions hold in the limited space. Thus the inequality below must be hold:

$$x \leqslant \left(\frac{1}{c_3}+V\right)t \qquad \text{for} \quad x \geqslant 0 \tag{86}$$

$$x \geqslant -\left(\frac{1}{c_3}-V\right)t \qquad \text{for} \quad x < 0 \tag{87}$$

relation (86) and (87) give the approximate ranges of validity of solutions. If the points out of the approximate range of validity of solutions the displacements vanish and this region remains undisturbed.

Here we consider some limited case:

(I) The source is motionless, but it is switched on at $t = 0$. This corresponds to $V = 0$ in equation (23). Obviously the solutions above cannot satisfy the case when $V = 0$. In this case Eqs. (67) and (68) degenerate into equations below:

$$u^*|_{y=-h} \sim \frac{2iQ_0\xi(1+p\tau_1)(e^{-a_1h}-e^{-a_2h})}{p^4(L_1^2+4\xi^2L_2^2)} \tag{88}$$

$$v^*|_{y=-h} \sim \frac{Q_0(1+p\tau_1)(e^{-a_1h}-e^{-a_2h})}{p^3(L_1^2+4\xi^2L_2^2)} \tag{89}$$

taking inverse Laplace and Fourier transform from Eqs. (88) and (89), gives

$$u|_{y=-h} \approx \frac{\sqrt{2\pi}\, Q_0}{24L_2^2}\, [e^{-c_2 h} f_{13} H(t-c_1 h) - e^{-c_4 h} f_{14} H(t-c_3 h)] \tag{90}$$

$$v|_{y=-h} \approx \frac{\sqrt{2\pi}\, Q_0}{8L_1 L_2}\, [e^{-c_2 h} H(t-c_1 h)\, f_{15} - e^{-c_4 h} H(t-c_3 h)\, f_{16}] \tag{91}$$

Obviously the results coincide with that obtained when $V \to 0$.

(II) The relaxation times $\tau_1 \to 0$, $\tau_2 \to 0$ the conventional coupled thermoelastic theory will be obtained. Here the displacements in transforms domain can be obtained as below:

$$u^*|_{y=-h} \sim \frac{2iQ_0 \xi \beta^5 (e^{-a_1 h} - e^{-a_2 h})}{p^4(\beta^3 + 4\xi^2)} \tag{92}$$

$$v^*|_{y=-h} \sim \frac{Q_0(1+p\tau_1)\, \beta^5 (e^{-a_1 h} - e^{-a_2 h})}{p^3(\beta^3 + 4\xi^2)} \tag{93}$$

taking inverse Laplace and Fourier transforms from Eqs. (92) and (93), gives

$$u|_{y=-h} \approx \frac{\sqrt{2\pi}\, Q_0 \beta^5}{24} \left[e^{-\frac{h\beta\varepsilon}{2}} f_{17} H\left(t - \frac{h}{\beta}\right) - f_{18} \right] \tag{94}$$

$$v|_{y=-h} \approx \frac{\sqrt{2\pi}\, Q_0 \beta^3 \sqrt{\beta}}{8} \left[e^{-\frac{h\beta\varepsilon}{2}} H\left(t - \frac{h}{\beta}\right) f_{19} - f_{20} \right] \tag{95}$$

where

$$f_{17} = \left(t - \frac{h}{\beta}\right)^3 \left[-\cosh\left(\frac{x\beta\sqrt{\beta}}{2}\right) \mathrm{Sign}[x] + \sinh\left(\frac{x\beta\sqrt{\beta}}{2}\right) \right] \tag{96}$$

$$\begin{aligned} f_{18} &= \frac{1}{120}\left[\cosh\left(\frac{x\beta\sqrt{\beta}}{2}\right) \mathrm{Sign}[x] - \sinh\left(\frac{x\beta\sqrt{\beta}}{2}\right) \right] \\ &\times \left\{ \frac{2h\sqrt{t}}{\sqrt{\pi}} (h^4 + 28h^2 t + 132t^2)\, e^{-\frac{h^2}{4t}} \right. \\ &\left. - \left[1 - \mathrm{erf}\left(\frac{h}{2\sqrt{t}}\right) \right] (h^6 + 30h^4 t + 180h^2 t^2 + 120t^3) \right\} \end{aligned} \tag{97}$$

$$f_{19}=\left(t-\frac{h}{\beta}\right)^{2}\left[-\cosh\left(\frac{x\beta\sqrt{\beta}}{2}\right)+\sinh\left(\frac{x\beta\sqrt{\beta}}{2}\right)\mathrm{Sign}[x]\right] \tag{98}$$

$$f_{20}=\frac{1}{12}\left[-\cosh\left(\frac{x\beta\sqrt{\beta}}{2}\right)+\sinh\left(\frac{x\beta\sqrt{\beta}}{2}\right)\mathrm{Sign}[x]\right]$$
$$\times\left\{-\frac{2h\sqrt{t}}{\sqrt{\pi}}(h^2+10t)\,e^{-\frac{h^2}{4t}}+\left[1-\mathrm{erf}\left(\frac{h}{2\sqrt{t}}\right)\right](h^4+12h^2t+12t^2)\right\} \tag{99}$$

Due to the presence of the error functions in the expressions of f_{18} and f_{20}, we can conclude that the conventional thermoelastic theory predicts an infinite velocity of heat propagation.

6. NUMERICAL RESULTS

Consider the material medium as that of copper. The parameters below are used:

$$\lambda=7.55\times10^{10}\ \mathrm{kg\cdot m^{-1}\cdot s^{-2}},$$
$$C_e=3.845\times10^{2}\ \mathrm{m^{2}\cdot K^{-1}\cdot s^{-2}}$$
$$\mu=3.86\times10^{10}\ \mathrm{kg\cdot m^{-1}\cdot s^{-2}},$$
$$a_t=17.87\times10^{-6}\cdot\ \mathrm{K^{-1}},$$
$$\rho=8.96\times10^{3}\ \mathrm{kg\cdot m^{-3}},$$
$$\theta_0=300\ \mathrm{K},$$
$$k=3.98\times10^{2}\ \mathrm{kg\cdot K^{-1}\cdot m\cdot s^{-3}},$$
$$\tau_1=0.5363\times10^{-12}\ \mathrm{s},$$
$$\tau_2=0.4348\times10^{-12}\ \mathrm{s}.$$

Thus the nondimensional quantities in the present analysis will be $\beta=1.98896$, $\varepsilon=0.0168$, $\tau_1=0.02$, $\tau_2=0.016214$ and we take the non-dimensional quantities $h=0.2$, $V=0.15$, $Q_0=0.1$.

At the boundary surface, $y=-h$, the distributions of displacements versus time are shown in Figs. 1 and 3 at different points $x=-0.6, 0.0, 0.6$. Figure 1 shows the x-component of the displacement solution. We can see that the displacement increases monotonically with t for $x\geqslant0$ and decrease monotonically with t for $x<0$ at small time range. The displacement attains its maximum value at the point $x=0.0$. Due to the existence of the step function in the expression of displacements at small time range, the displacements remain undisturbed for time $t\leqslant c_3h$ (0.0254). Figure 3 shows

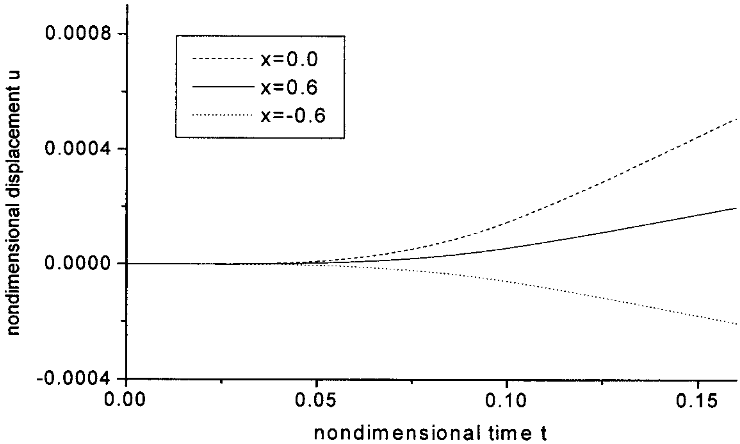

Fig. 1. Displacment at different points at the boundary versus time.

the y-component of the displacement solution. We can see that the displacement increases monotonically with t for all points and attains its maximum value at location $x = 0.0$. Figures 2 and 4 depict that the displacements profile at the boundary at different time, we can see the displacements always starts form the zero value and terminates at the zero value. Also we can see that out of the approximate range of validity of solutions (e.g., for $t = 0.05$, the approximate range is $-0.39 \leqslant x \leqslant 0.4$) the displacements vanish and this region remains undisturbed.

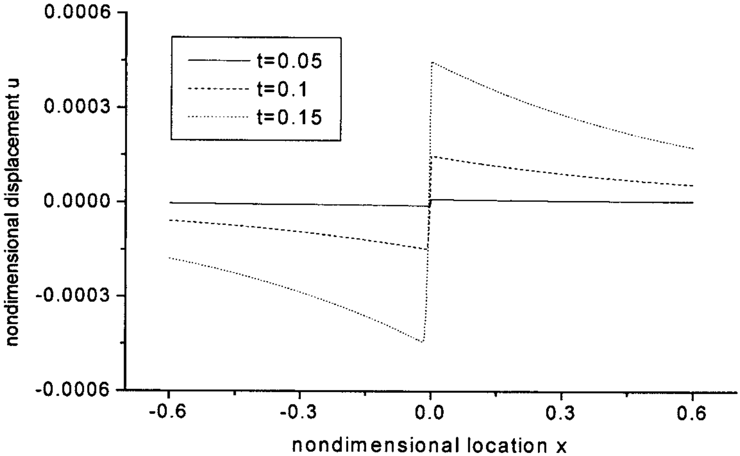

Fig. 2. Profile of displacement at the boundary at different time.

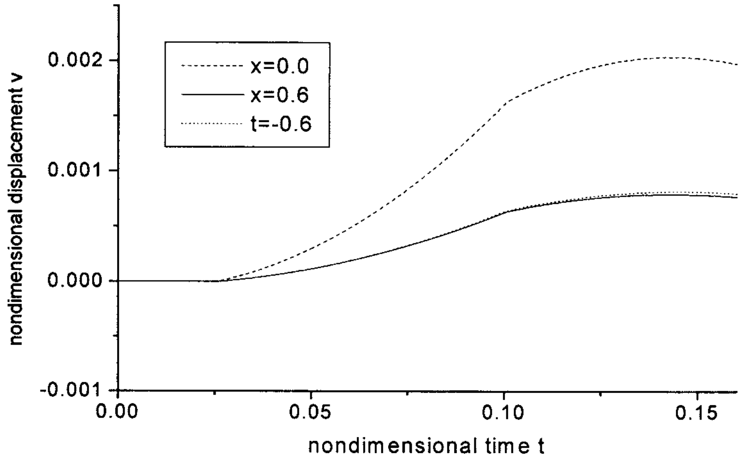

Fig. 3. Displacment at different points at the boundary versus time.

Figures 5 and 6 give the profile of displacement at the boundary at $t = 0.1$ for GL theory and conventional theory. We can see that the relaxation times have salient effect to the distribution of displacement at small time range. The two figures show that the relaxation times make the displacement smaller than the displacements corresponding to the conventional theory. With the increase of time t, the effect of relaxation time will decrease. So if the considered time is much longer than the relaxation times, the conventional thermoelastic theory can describe the problem exactly.

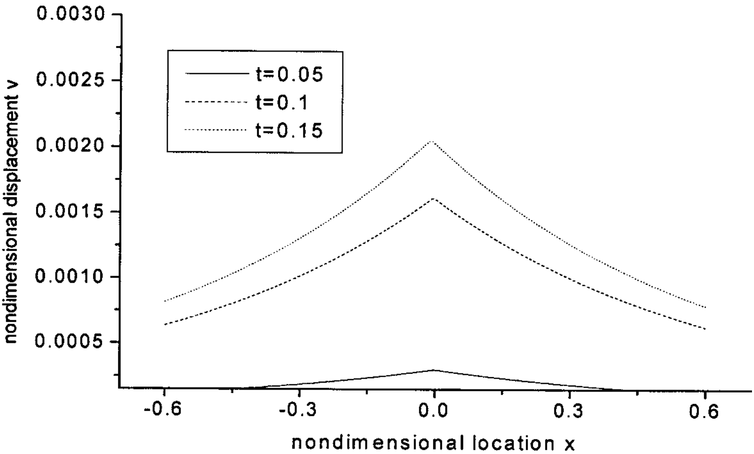

Fig. 4. Profile of displacement at the boundary at different time.

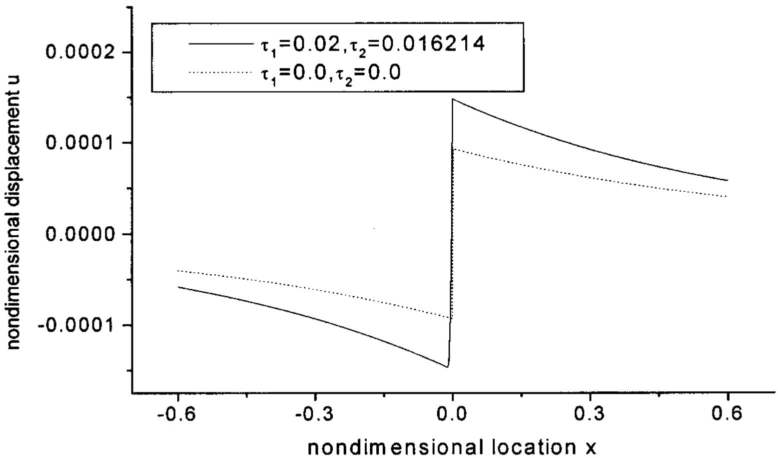

Fig. 5. Profile of displacement at the boundary at $t = 0.1$ for GL theory and conventional theory.

Figures 7 and 8 compare the displacements by not keeping ξ in Eqs. (63)–(65) with those obtained by keeping ξ in Eqs. (63)–(65). We can see that the effect of ξ is very tiny. So we only consider two terms in Eqs. (63)–(64) and one term in Eq. (65). This will greatly reduce the calculation time.

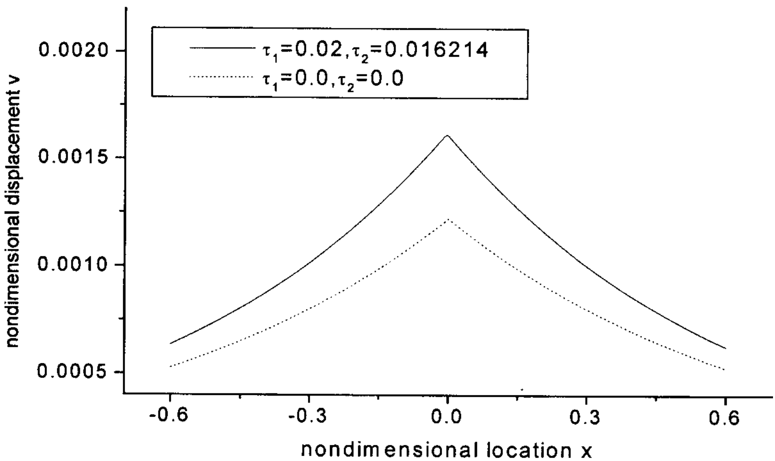

Fig. 6. Profile of displacement at the boundary at $t = 0.1$ for GL theory and conventional theory.

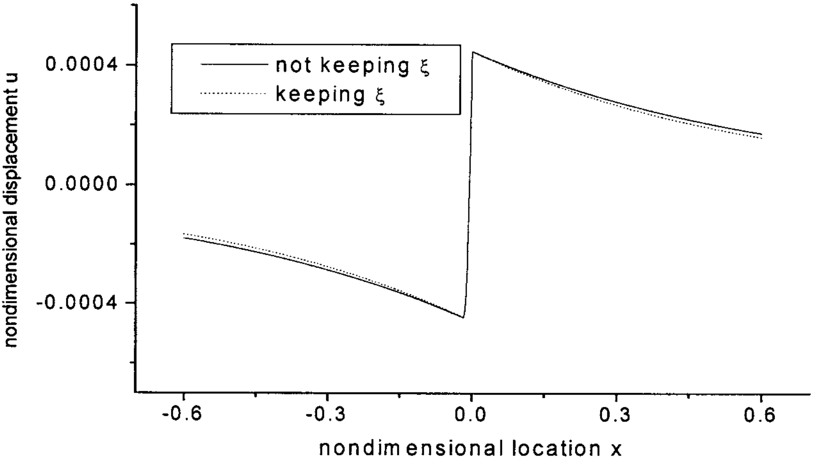

Fig. 7. Effect of the small terms on the displacement at time $t = 0.15$.

7. CONCLUSION

Transient waves created by a line heat source that suddenly starts moving with a uniform velocity inside isotropic homogeneous thermoelastic half-space are studied with thermal relaxation of the type of Green and Lindsay. The problem is reduced to the solution of three differential equations, one involving the elastic vector potential, and the other two coupled, involving the thermoelastic scalar potential and the temperature.

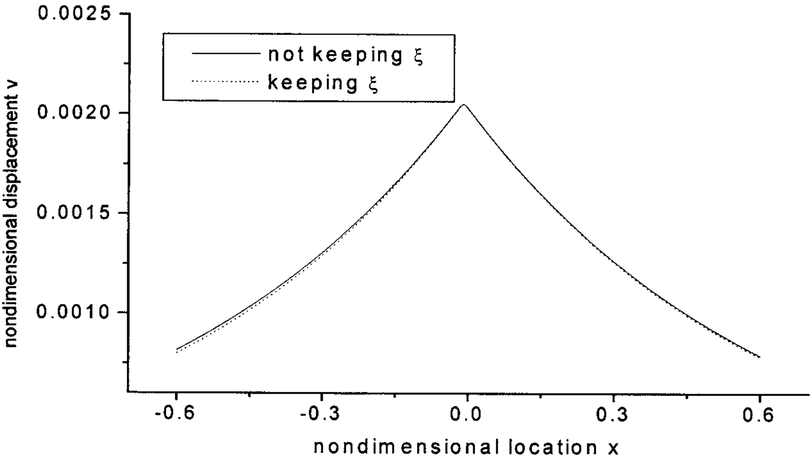

Fig. 8. Effect of the small terms on the displacement at time $t = 0.15$.

Using joint Laplace and Fourier transforms the problem is solved. The expression for displacements valid in the small time range are obtained in transforms domain and the displacements are calculated at the boundary by using inverse transforms for small time. Also the displacements in the transform domain indicate the existence of dilatational, transverse and thermalelastic waves inside the medium and the velocities of the three kinds of waves are given in this paper. The approximate region valid for the solution is given and two special cases are considered. Also the results are graphically described for the medium of copper. The results show that the relaxation times have salient effect to the distribution of displacement at small time range.

ACKNOWLEDGMENTS

This work was supported by Grants 10132010 and 50175089 from National Nature Foundation of PRC and the Doctoral Foundation of Xi'an Jiaotong University. We also thank the reviewers for their valuable comments which led to the improvement of the paper.

REFERENCES

1. H. W. Lord and Y. Shulman, *J. Mech. Phys. Solid.* **15**:299 (1967).
2. A. E. Green and K. E. Lindsay, *J. Elasticity* **2**:1 (1972).
3. D. S. Chandrasekharaiah, *Appl. Mech. Rev.* **39**:355 (1986).
4. D. S. Chandrasekharaiah, *Appl. Mech. Rev.* **51**:705 (1998).
5. Sarbani Chakravorty and Amitava Chakravorty, *Int. J. Math. Math. Sci.* **21**:595 (1998).
6. D. S. Chandrasekharaiah and K. R. Srikantiah, *India J. Tech.* **25**:1 (1987).
7. D. S. Chandrasekharaiah and H. N. Murthy, *Acta Mech.* **89**:1 (1991).
8. J. Ignaczak and E. Mrowka-Matejewska, *J. Thermal Stresses* **13**:281 (1990).
9. R. B. Hetnarski and J. Ignaczak, *J. Thermal Stresses* **16**:473 (1993).